Maximum Information Bounds for Planning Active Sensing Trajectories

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***Abstract*— This paper considers the problem of planning trajectories for robots equipped with sensors whose task is to track an evolving target process in the world. We focus on processes which can be represented by a Gaussian random variable, which is known to reduce the general stochastic information acquisition problem to a deterministic problem, which is much simpler to solve. Previous work on solving the resulting deterministic problem focuses on computing a search tree by Forward Value Iteration and pruning uninformative nodes early on in the search via a domination criteria. In this work we formulate the Active Information Acquisition problem as a deterministic planning problem where algorithms like Dijkstra and A* can produce optimal solutions. To use A* effectively in long planning horizons we derive a consistent and admissible heuristic as a function of the sensor model which can be used in information acquisition tasks such as actively mapping static and moving targets in an environment with obstacles. We validate the results in several simulations indicating that the resulting heuristic informed algorithm can recover optimal solutions faster than existing search-based methods.**

I. Introduction

The problem of Active Information Acquisition has received significant attention in the robotics community due to its breadth of applications in localization and mapping [1]–[4], active target tracking [5], security and surveillance [6], and environmental monitoring [7], [8]. The core problem is to plan trajectories for one or more robots equipped with a sensor suite, such that some measure of information is optimized. Common formulations of this problem choose to maximize information theoretic quantities such as mutual information, or to minimize uncertainty quantities related to entropy.

We consider a non-myopic single robot information acquisition problem where the goal is to track the evolution of a multivariate Gaussian distribution for a fixed finite-time horizon. We show how this problem is an instance of a path planning problem, for which existing search and sampling based algorithms apply. The search-based family of algorithms includes the A* algorithm, a widely known algorithm for computing shortest paths in planning problems. The A* algorithm depends critically on having access to a heuristic function which can approximate the remaining cost-to-go of a state in the search tree. We propose the first heuristic for planning in information acquisition problems, and prove its consistency and admissibility, which guarantee that A* returns an optimal solution to the planning problem. We demonstrate in several simulations that the usage of this heuristic drastically reduces the size of the search tree in comparison to other methods.

Related Work. There are many formulations of the active information acquisition problem that differ in one or more dimensions: size of the robot team (single [9] or multi-robot [10]), length of the planning horizon (greedy [11], non-myopic [12], [13], infinite horizon [14]), the class of target probability distribution (Gaussian [9], Gaussian Mixture [15], Particle [16]), or the planning approach (search-based [9], [12], sampling based [17]). As the problem becomes more complex through the addition of robots, a longer horizon or a more general probability distribution, sampling based methods become more attractive, though they only provide asymptotic optimality guarantees. On the other hand, search based methods can obtain (sub)-optimality guarantees in problems with linear assumptions [12] or submodular cost functions [18]. There also exist optimization based methods which can handle continuous action spaces, but are only locally optimal [19].

Common to both search and sampling-based planning are the concepts of pruning and heuristic-guided search. Pruning based on branch and bound techniques has been applied in [20], and domination-based pruning has been considered in [21]. In contrast, heuristic-based search has been neglected in information acquisition problems due to the difficulty of constructing informative heuristics. A heuristic can loosely be defined as a function that estimates how promising a state is in a planning problem.

In this work, we cast the Linear Gaussian Information Acquisition problem as a search-based planning problem, which allows us to use well-known algorithms such as Dijkstra and A* search. To effectively utilize the A* algorithm, which is a best-first search method that uses a heuristic to estimate the cost-to-go until the goal region, we propose a heuristic and prove its consistency and admissibility. Thus using this heuristic for A* search enables us to recover the optimal solution to the information acquisition problem. The derived heuristic depends on upper bounds of the sensor information matrix that comes from the Information Filter form of the Kalman filter. We derive the necessary upper bounds for a variety of sensor types, and demonstrate the overall effectiveness of our planning approach in comparison to existing search-based approaches that prune based on domination criteria.

Contributions. Previous work has shown that the general Active Information Acquisition problem is a deterministic

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optimal control problem under Linear and Gaussian assumptions, and provided efficient methods to prune an exhaustive forward search tree. In this paper:

1) We propose the *first* consistent heuristic function for information acquisition problems, and define it in terms of upper bounds on the Sensor Information Matrix.
2) We derive the necessary upper bounds on the Sensor Information Matrix for the commonly used position, range, bearing, and camera sensing models.
3) We provide simulation results using an A* algorithm with our proposed heuristic, and show it can generate optimal solutions to the planning problem in less time than existing approaches.

II. Problem Formulation

Consider a mobile sensing robot, with discrete-time dynamics described by the following motion model:

$$x_{t+1} = f(x_t, u_t), \tag{1}$$

where $x_t \in \mathcal{X} \cong \mathbb{R}^{n_x}$ is the n_x-dimensional state of the robot at time t, with metric $d_{\mathcal{X}}$, $u_t \in \mathcal{U}$ is the control action applied to the robot at time t, and the set $\mathcal{U}$ of possible control inputs for robot i is finite. The goal of the robots is to track the evolution of a target process with (unknown) state y_t and dynamics:

$$y_{t+1} = a(y_t) + w_t, \quad w_t \sim \mathcal{N}(0, W_t) \tag{2}$$

where $y_t \in \mathbb{R}^{n_y}$, and w_t is Gaussian with covariance W_t, and independent across all timesteps. We also define a metric $d_{xy}(x, y) : \mathcal{X} \times \mathbb{R}^{ny} \to \mathbb{R}^+$.

The operation of each sensor is described by the following *sensor observation model*:

$$z_t = c(x_t, y_t) + v_t(x_t), \tag{3}$$

$$v_t(x_t) \sim \mathcal{N}(0, V_t(x_t)), \tag{4}$$

where $z_t \in \mathbb{R}^{d_{z_i}}$ is the measurement obtained by the robot at time t, and $v_t(x_t)$ is a sensor-state-dependent Gaussian noise, whose values are independent at any pair of times. Here we note that the sensor model is allowed to be a general nonlinear function of x_t and y_t.

The information available to the robot at time t is denoted:

$$\mathcal{I}_t := (z_{0:t}, u_{0:(t-1)}), t > 0 \tag{5}$$

It was shown in [12] that if both the target process dynamics and sensor observation model are linear in the target state, that is $y_{t+1} = Ay_t + w_t$, and $z_{t+1} = C(x_t)y_t + v_t$, a separation principle holds and the stochastic optimal control problem can be reduced to a deterministic optimal control problem.

Problem 1 (Deterministic Information Acquisition)**.** *Given an initial sensor state $x_0 \in \mathcal{X}$, a Gaussian prior distribution of the target states (y_0, Σ_0), and a finite planning horizon T, choose a sequence of control inputs $u_{0:T-1} \in \mathcal{U}$ for $t = 1, \ldots, T$, which minimize the sum of stage costs c:*

$$\begin{aligned} \underset{u_{0:u_{T-1}}}{min} \; J_T &= \sum_{t=0}^{T-1} \log\det \Sigma_t \qquad (6)\\ \textit{s.t.} \quad x_{t+1} &= f(x_t, u_t), \qquad t = 0, ..., T-1 \\ \Sigma_{t+1} &= \rho^e_{x_{t+1}}(\rho^p_t(\Sigma_t)), \;\; t = 0, ..., T-1 \end{aligned}$$

where $\rho^e_x(\Sigma)$ is the Kalman filter measurement update, and ρ^p_t is the Kalman filter prediction step, as follows:

$$\begin{aligned} \textbf{\textit{Predict:}} \quad & \rho^p_t(\Sigma) := A_t \Sigma A_t^T + W_t \\ \textbf{\textit{Update:}} \quad & \rho^e_x(\Sigma) := (\Sigma^{-1} + M(x))^{-1} = F_x(\Sigma)\Sigma \\ & M(x) := C(x)^T V(x)^{-1} C(x) \\ & F_x(\Sigma) := I - K_x(\Sigma)C(x) \\ & K_x(\Sigma) := \Sigma C(x)^T R_x^{-1}(\Sigma) \\ & R_x(\Sigma) := C(x)\Sigma C(x)^T + V \end{aligned}$$

where $M(\cdot) \in \mathbb{R}^{n_y \times n_y}$ is called the sensor information matrix.

We note here that although the linear models may appear restricting, it is possible to apply nonlinear process and observation models by linearizing about a predicted target trajectory and considering the covariance predictions produced by the Extended Kalman Filter. This technique has been applied in information acquisition problems in [13], and we will also do so in this work.

III. Information Acquisition as a Deterministic Shortest Path

In this work, we view the information acquisition optimal control problem as a planning problem. We state a standard definition of a planning problem for clarity:

Problem 2 (Planning Problem)**.** *Given a set of states S, an initial state $s_0 \in S$, a boolean function $G : S \to \{0, 1\}$ which tells us whether the state is in the goal region, a function $A(s)$ which produces the valid control inputs in a given state s, a transition function $T(s, a) : S \times A(s) \to S$ for $s \in S$ and $a \in A(s)$, and a function $c(s, s') : S \times S \to \mathbb{R}$ returning the cost of being in state s and transitioning to state s', find a path $P = \{s_0, \ldots, s_n\}$:*

$$\underset{P}{min} \quad J(P) = \sum_P c(s, s') \tag{7}$$

$$\textit{s.t.} \;\; s_{i+1} = T(s_i, a_i) \tag{8}$$

$$G(s_n) = 1 \tag{9}$$

We now define the information acquisition problem as an instance of the planning problem. We define the planning state space to include both the spatial state and the information state of the information acquisition problem, in addition to the number of elapsed time steps since the initial time:

$$S := \{(x_t, \Sigma_t, t) \mid x_t \subseteq \mathcal{X}, \Sigma_t \succeq 0, t \geq 0\}$$

Then we can let the initial state s_0 be defined:

$$s_0 := (x_0, \Sigma_0, 0)$$

In the classic planning problem, the length of the path is unknown a priori and is determined as part of the search procedure. In this problem, we focus on a fixed horizon control problem, so our goal function can be defined:

$$G(s) := G(x_t, \Sigma_t, t) = \begin{cases} 1 & \text{if } t = T \\ 0 & \text{otherwise} \end{cases}$$

Next, we can define the available actions at any state to be the set of actions that give collision-free paths.

$$A(s) := A(x_t, \Sigma_t, t) = \{u \mid u \in \mathcal{U}\}$$

We also define the transition function that allows us to evaluate the next state given the current state and action:

$$\begin{aligned} T(s,a) :=& T(x_t, \Sigma_t, t, u_t) \\ =& [f(x_t, u_t), \rho^e_{f(x_t,u_t)}(\rho^p_t(\Sigma_t)), t+1] \end{aligned}$$

Finally, we define the state cost function to be the log determinant of the target covariance matrix:

$$c(s,a) := c(x_t, \Sigma_t, t, u_t) = \log\det \Sigma_{t+1}.$$

Deterministic shortest path problems such as Problem 2 can be solved using search or sampling-based methods. For example, A* is a best-first graph-search algorithm that works by expanding the most promising nodes in a search tree beginning at a root state. Let g to be the cost incurred to reach the state (x_k, Σ_t, t) from the root:

$$g(x_t, \Sigma_t, t) = \sum_{k=0}^{t-1} \log\det \Sigma_k. \tag{10}$$

Next, we define a heuristic function $h(x_t, \Sigma_t, t)$, which serves as an estimate of the remaining cost-to-go along the path going through the state (x_t, Σ_t, t). An *informative* heuristic function massively improves exploration in a planning problem by delaying or ruling out regions of the graph which appear to be unpromising. If a planner has access to an optimal heuristic h^*, that is a heuristic which exactly equals the lowest possible cost-to-go from the desired state to the goal region, the planner will explore the exact optimal path and result in a linear time search. On the other hand, a trivial heuristic can be defined by $h(\cdot) = 0$. This would be considered an uninformative heuristic, since it provides no information about how promising or unpromising the current state is. Moving forward, our goal will be to design efficient and informative heuristic functions for the information gathering problem, since this will help remove unpromising regions from the search space.

A* is typically implemented with a priority queue, *OPEN*, where the order of state expansion is determined by the element in the priority queue with minimal p-value, which is the sum of the cost-to-come and the cost-to-go:

$$p(x_t, \Sigma_t, t) = \underbrace{g(x_t, \Sigma_t, t)}_{\text{cost-to-come}} + \underbrace{h(x_t, \Sigma_t, t)}_{\text{cost-to-go}}$$

Applying the algorithm above yields an optimal solution, provided the following two conditions hold on the heuristic function $h(\cdot)$:

Algorithm 1 A* Algorithm (x_0, Σ_0, T, $\mathcal{X}_g$)

g$(x_0, \Sigma_0, 0)$ = 0; OPEN = $\emptyset$
Insert $(x_0, \Sigma_0, 0)$ into OPEN with $p(x_0, \Sigma_0, 0)$ = h$(x_0, \Sigma_0, 0)$
$(x_t, \Sigma_t, t) \leftarrow \underset{(x_t,\Sigma_t,t)\in OPEN}{\arg\min}\ p(x_t, \Sigma_t, t)$
while $(x_t, \Sigma_t, t) \notin \mathcal{X}_g$ **do**
  for For $u \in \mathcal{U}$ **do**
    $(x, \Sigma, k) = (f(x_t, u), \rho^e_{f(x_t,u)}(\rho^p_t(\Sigma_t)), t+1)$
    $g(x, \Sigma, k) = g(x) + \log\det \Sigma_t$
    $p(x, \Sigma, k) = g(x, \Sigma, k) + h(x, \Sigma, k)$
    Insert (x, Σ, k) into OPEN with $p(x, \Sigma, k)$
  $(x, \Sigma, t) \leftarrow \underset{(x,\Sigma,t)\in OPEN}{\arg\min}\ p(x, \Sigma, k)$

- Admissibility: $h(s) \leq h^*(s) \ \ \forall s \in S$
- Consistency: $h(s) \leq c(s, s') + h(s') \ \ \forall s \in S$

IV. Information Acquisition Heuristics

While the method detailed in Algorithm 1 is optimal, without an informative heuristic, the algorithm will default to Dijkstra's algorithm and will require the expansion of all nodes, causing the search complexity grow exponentially as $\mathcal{O}(|\mathcal{U}|^T)$. An informative heuristic dramatically speeds up the planning procedure, so we now focus our attention on deriving heuristics for the information acquisition problem.

A. Reduction to Bounding Sensor Information Matrix

A key observation we make in the problem structure is the relationship of the cost function to the Kalman filter covariance matrix $\Sigma_t \succeq 0$, and the sensor information matrix $M(x) = H(x)^T V(x)^{-1} H(x) \succeq 0$. Our objective is to reduce the computation of a heuristic to computation of an upper bound on the sensor information matrix over the reachable state space of the sensor.

Definition 1 (Reachable-Set)**.** *Given an initial sensor state x_0, the t-step reachable set can be defined for $t > 0$ as:*

$$\mathcal{R}^t(x_0) := \{x' \mid x' = f(x, u) \ \ \forall u \in \mathcal{U} \ \ \forall x \in R^{t-1}(x_0)\}$$

where $\mathcal{R}^0(x_0) := \{x_0\}$.

Lemma 1. *Given a prior covariance matrix of a Gaussian distribution $\Sigma \succeq 0$, the Kalman filter prediction step $\rho^p_t(\Sigma) \succeq 0$. Moreover, given $0 \preceq M(x) \preceq \bar{M}^1(x)$, the following holds:*

$$\begin{aligned} &\rho^p_t(\Sigma_t) + M(x_t) \preceq \rho^p_t(\Sigma_t) + \bar{M}^1(x_t) \\ &(\rho^p_t(\Sigma_t) + \bar{M}^1(x_t))^{-1} \preceq (\rho^p_t(\Sigma_t) + M(x_t))^{-1} \end{aligned}$$

Theorem 1 (Heuristic Functions for Information Acquisition)**.** *Let the reachable set by the robot at state x in $t > 0$ timesteps be denoted by $\mathcal{R}^t(x)$. Suppose there exists a matrix $\bar{M}^t(\bar{x})$ such that $\bar{M}^t(\bar{x}) \succeq M(\bar{x}) \quad \forall \bar{x} \in \mathcal{R}^t(x)$. Then the following heuristic is consistent and admissible:*

$$h(x_t, \Sigma_t, t) = \sum_{k=t}^{T} \log\det((\rho^p_k(\Sigma_k)^{-1} + \bar{M}^{k-t+1}(x_t))^{-1}),$$

where $\Sigma_{k+1} = (\Sigma_k^{-1} + \bar{M}^{k-t+1}(x_t))^{-1}$ and $(x_k, \Sigma_k, k) = (x_t, \Sigma_t, t)$

Proof. In [22], it is shown that consistency implies admissibility for heuristic functions. Therefore, we only need to show the proposed heuristic is consistent, which requires the following

$$h(x_t, \Sigma_t, t) \leq \log\det(\Sigma_{t+1}) + h(x_{t+1}, \Sigma_{t+1}, t+1)$$

We have:

$$\begin{aligned}
&h(x_t, \Sigma_t, t) - h(x_{t+1}, \Sigma_{t+1}, t+1) \leq \log\det(\Sigma_{t+1}) \\
&= \sum_{k=t}^{T} \log\det((\rho_k^p(\Sigma_k)^{-1} + \bar{M}^{k-t+1}(x_t))^{-1}) \\
&- \sum_{k=t+1}^{T} \log\det((\rho_{k+1}^p(\Sigma_{k+1})^{-1} + M^{k-t+1}(x_t))^{-1}) \\
&= \log\det((\rho_t^p(\Sigma_t)^{-1} + \bar{M}^1(x_t))^{-1}) \\
&\underset{a}{\leq} \log\det((\rho_t^p(\Sigma_t)^{-1} + M(x_t))^{-1}) \\
&\underset{b}{=} \log\det(\Sigma_{t+1})
\end{aligned}$$

Where (a) holds by monotonicity of $\log\det(\cdot)$, and by Lemma 1 since $M(x) \preceq \bar{M}^1(x)$. Then (b) holds by the Kalman Filter Riccati Map. □

Corollary 1 (Optimality and Inflated Heuristic)**.** *The solution obtained by Algorithm 1 with the heuristic function proposed in Theorem 1 returns an optimal solution J_T^*. If the heuristic is scaled by a factor of ϵ, the returned solution J_ϵ has bounded sub-optimality such that $J_T^* \leq J_\epsilon \leq \epsilon J_T^*$.*

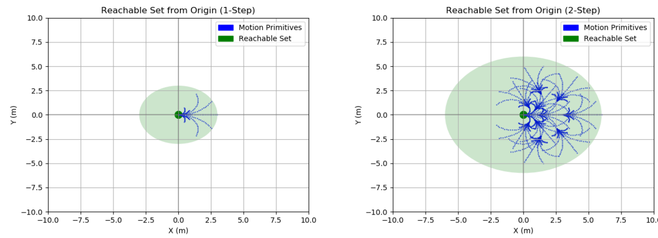
Proof. The proof follows immediately from the consistency and admissibility properties proved in Theorem 1, and from the bounds on inflated heuristics obtained in [23]. □

Remark. The result obtained in Theorem 1 holds for any monotone cost function of the covariance Σ_t, and is not restricted to $\log\det(\Sigma_t)$. For instance, other commonly used uncertainty measures such as the trace $tr(\Sigma_t)$ also lead to consistent heuristics.

B. Approximation of the Reachable Set

The implication of Theorem 1 is two-fold. First, it constructs a consistent, and admissible heuristic function that can be used to speed up the planning process. Second, it reduces this heuristic computation to the problem of computing an upper bound on the sensor information matrix over a reachable set $\mathcal{R}^t$, i.e. $M(x) \preceq \bar{M}^t(x) \quad \forall x \in \mathcal{R}^t$.

A first step towards computing an upper bound on the sensor information matrix is to approximate the t-step reachable set $\mathcal{R}^t(x)$ from a given sensor state x. Because we consider finite action spaces, the true reachable set from a given configuration grows exponentially in the number of timesteps. Though there are more intricate methods to compute the reachable set of a dynamical system as detailed in [24], we resort to an over-approximation by a ball of finite



(a) Reachable set $\mathcal{R}^1(0,0)$. (b) Reachable set $\mathcal{R}^2(0,0)$.

Fig. 1: Here we show the evolution of a finite set of motion primitives, and the approximate reachable set computation. Figure (a) shows one timestep, and Figure (b) shows two timesteps.

radius $r_{\mathcal{U}}(x)$, with an increasing radius in the number of timesteps for its simplicity.

$$r_{\mathcal{U}}(x) = \max_{u \in \mathcal{U}} d_{\mathcal{X}}(x, f(x,u)) \tag{11}$$

Thus, an over-approximation $\bar{\mathcal{R}}^t(x) \supseteq \mathcal{R}^t(x)$ for the set reachable in t timesteps can be constructed as:

$$\bar{\mathcal{R}}^t(x) := \{\bar{x} \in \mathcal{X} \mid d_{\mathcal{X}}(\bar{x}, x) \leq t * r_{\mathcal{U}}(x)\} \tag{12}$$

Note that for this to be an over-approximation, some minor continuity assumptions must hold on the motion model. . See Figure 1 to visualize the construction of the approximate reachable set $\bar{\mathcal{R}}^t(x)$

Given a construction of a reachable set, we focus on the problem of computing a bounding sensor information matrix such that $M(\bar{x}) \preceq \bar{M}^t(\bar{x}) \ \forall \bar{x} \in \bar{\mathcal{R}}^t(x)$. Doing this requires examination of specific sensor models.

V. Bounding the Sensor Information Matrix

In this section, we derive bounding sensor information matrices for several common sensors that are often used in essential robotics tasks such as localization and mapping problems, target tracking, and others. We begin by extending the reachable set concept to account for observability:

Definition 2 (Reachable-Observable Set)**.** *Given an initial sensor state x_0, a t-step reachable set $\mathcal{R}^t(x_0)$, and a function $O(x,y) : \mathcal{X} \times \mathbb{R}^{n_y} \to \{0,1\}$ indicating whether the target process y is observable from a sensor state x, we can define the reachable-observable set as:*

$$\mathcal{O}^t(\mathcal{R}^t(x_0)) := \{y \mid O(x,y) = 1 \ \forall x \in \mathcal{R}^t(x_0)\} \tag{13}$$

The reachable-observable set includes all target states y which the sensor may observe in t-steps. Since we are only seeking upper bounds on the sensor information matrix, we can resort to an over-approximation of the reachable-observable set, similar to the method for approximating the reachable set itself. We assume here that a sensor has a maximum range for which it can observe a component of the target from $\bar{r}(y)$:

$$\begin{aligned}
\bar{r}(y) = &\max_{x \in \mathcal{X}} d_{xy}(x,y) \\
&\text{s.t. } O(x,y) = 1
\end{aligned}$$

Now, to over-approximate the reachable-observable set, we can build on top of the reachable set construction from Equation 12 as follows:

$$\bar{\mathcal{O}}^t(x) := \{y \mid O(x,y) = 1 \ \forall x \in \bar{\mathcal{R}}^t(x) \ \} \tag{14}$$

We now have all the machinery to derive bounds on the sensor information matrix for a variety of sensor types.

A. Unobservable Case

A simple but important case to begin with is when $y \notin \bar{\mathcal{O}}^t(x)$, that is a target is strictly non-observable in t-steps from a given sensor location. There is no possibility of obtaining a measurement in this case, the Kalman Filter resorts to a prediction only step. An alternate way to view this, is that the observation matrix $C(x)$ from sensor state x is simply zero. Since taking $C(x) = 0$ implies $M(x) = C(x)^T V^{-1}(x) C(x) = 0$, a trivial upper bound can be obtained as:

$$\bar{M}(x) = 0$$

when $y \notin \bar{\mathcal{O}}^t(x)$. The remaining sections now assume the target y is observable.

B. Position-Sensor

The position sensor reports the relative position of a point $y \in \mathbb{R}^3$ from a sensing location $x = (p, R)$ with position $p \in \mathbb{R}^3$, and orientation $R \in SO(3)$.

$$z = c(x,y) = R^T(y - p) + v, \quad v \sim \mathcal{N}(0, V(x)) \tag{15}$$

In the case where the point y is observable, the sensor observes a noisy estimate of the translated state $y - p$, where $C(x) = R^T$ is the orientation of the sensor in the world frame. In this case, we have that $M(x) = RV(x)^{-1}R^T$. We can bound $V(x)^{-1}$ above by it's maximum eigenvalue, i.e. $\lambda_{max} I_3 \succeq V(x)^{-1}$. This yields a bound of $M(x) = \lambda_{max} RR^T = \lambda_{max} I_3$, since $RR^T = I_3$. Thus, we can derive the following bound on the sensor information matrix:

$$\bar{M}^t(\bar{x}) = \lambda_{max} I_3 \tag{16}$$

C. Range-Sensor

The range sensor reports the relative distance of a point $y \in \mathbb{R}^3$ from a sensing location $x = (p, R)$ with position $p \in \mathbb{R}^3$, and orientation $R \in SO(3)$.

$$z = c(x,y) = \|p - y\|_2 + v, \quad v \sim \mathcal{N}(0, \sigma_r^2) \tag{17}$$

The range sensor is a non-linear sensor model, which requires us to adapt our formulation. Previous work has approached this problem by linearizing and planning based on the Extended Kalman Filter (EKF) covariance, which we will consider here. The linearized observation model is:

$$\nabla_{y=\hat{y}} c(p,y) = C(x) = \frac{(\hat{y} - p)^T}{\|p - \hat{y}\|_2} \tag{18}$$

Here we note that the linearized $C(x)$ is a row vector, and the sensing noise covariance $V(x) = \sigma_r^2$ is a scalar due to the 1-dimensional measurement. Thus the sensor information matrix $M(x) = C(x)^T V^{-1}(x) C(x)$ is an outer product scaled by the inverse noise covariance σ_r^{-2}. This outer product has one eigenvalue $\lambda = \sigma_r^{-2}\|C(x)\|_2$, and the remaining eigenvalues are zero. Taking the norm of $C(x)$:

$$\|C(x)\| = \frac{1}{\|p - \hat{y}\|_2^2} (\hat{y} - p)^T (\hat{y} - p) \tag{19}$$

$$= \frac{\|\hat{y} - p\|_2^2}{\|\hat{y} - p\|_2^2} \tag{20}$$

$$= 1. \tag{21}$$

Thus we can upper bound the sensor information matrix as:

$$\bar{M}^t(\bar{x}) = \sigma_r^{-2} I_3 \tag{22}$$

D. Bearing Sensor

The bearing sensor reports the relative bearing of a point $y \in \mathbb{R}^2$ from a sensing location $x = (p, \theta)$ with position $p \in \mathbb{R}^2$, and orientation $\theta \in [-\pi, \pi]$.

$$z = c(x,y) = \tan^{-1}\left(\frac{y_2 - p_2}{y_1 - p_1}\right) - \theta + v, \quad v \sim \mathcal{N}(0, \sigma_b^2) \tag{23}$$

As for the range-sensor, we note that this is a non-linear sensor and must be linearized to obtain covariance estimates using the EKF. The linearization about the point y is given:

$$\nabla_{y=\hat{y}} c(x,y) = C(x) = \frac{1}{\|p - \hat{y}\|_2^2} \begin{bmatrix} -(\hat{y}_2 - p_2) & (\hat{y}_1 - p_1) \end{bmatrix} \tag{24}$$

Because the sensor is one dimensional with inverse noise covariance given by σ_b^{-2}, it can be easily seen that the sensor information matrix $M(x) = \frac{\sigma_b^{-2}}{\|p - \hat{y}\|_2}$, by applying the result from above for the range sensor. The difficulty in computing this bound is the need to place a limit on how close the sensor and target can be to avoid the singularity in $M(x)$, owing to the fact that there is an extra division by $\|p - y\|_2$. To avoid this singularity, suppose we place a range limit of $\underline{r}$ to denote the minimum range a measurement can be taken, similar to the maximum sensing range requirement. Then the following bound holds for $y \in \bar{\mathcal{O}}^t(\bar{x})$:

$$\bar{M}^t(\bar{x}) = \frac{\sigma_b^{-2}}{\max\{\underline{r}, \min_{p \in \bar{\mathcal{R}}^t(\bar{x})} \|p - \hat{y}\|_2\}} I_2 \tag{25}$$

E. Camera Sensor

Lastly, we consider the camera sensor, which reports the pixel location $z \in \mathbb{N}^2$ in image coordinates of a point $y \in \mathbb{R}^3$ in 3-D space, given a camera pose $x = (p, R)$ with $p \in \mathbb{R}^3$ and $R \in SO(3)$, and an intrinsic camera matrix $K \in \mathbb{R}^{2\times 3}$.

$$z = c(x,y) = K\pi(R^T(y - p) + v), \ \ v \sim \mathcal{N}(0, V) \tag{26}$$

where $\pi(y) := \frac{1}{y^3} y$ is a projection function.

We can take the gradient of the observation model with respect to a linearization point: $\hat{y}$:

$$\nabla_{y=\hat{y}} c(x,y) = C(x) = K\pi'(R^T(y - p))R^T \tag{27}$$

$$\pi'(x) = \frac{1}{x_3^2} \begin{bmatrix} x_3 & 0 & -x_1 \\ 0 & x_3 & -x_2 \\ 0 & 0 & 0 \end{bmatrix} \tag{28}$$

To simplify notation, we let $P = \pi'(R^T(y-p))$. The sensor matrix $M(x) = C(x)^T V(x)^{-1} C(x)$ can be written as:

$$M(x) = RP(x)^T K^T V(x)^{-1} KP(x) R^T \tag{29}$$

Let λ_{KV} be the maximum eigenvalue of the matrix $K^T V^{-1} K \succeq 0$. Then let λ_P be the maximum eigenvalue of the matrix $P^T P$.

$$M \preceq \lambda_{KV} \lambda_P R R^T \tag{30}$$

$$\preceq \lambda_{KV} \lambda_P I_3 \tag{31}$$

The eigenvalues of $P^T P$ are $\lambda = \{1/x_3^2, \|x\|_2^2/x_3^4, 0\}$, so the maximum is always $(x_1^2 + x_2^2 + x_3^2)/x_3^4$. The final bound can be expressed as:

$$\bar{M}^t(\bar{x}) = \lambda_{KV} I_3 \max_{p \in \bar{\mathcal{R}}^t(\bar{x})} \frac{\|y-p\|_2^2}{(e_3 R^T (y-p))^4} \tag{32}$$

If the minimum depth approaching zero causes a singularity in the above expression, bounds can be introduced similar to the bearing sensor to ensure the information matrix remains finite.

VI. Application to Active Mapping

A. Problem Setup

We apply the A* planning algorithm with the proposed information acquisition heuristic to an active mapping problem. We consider a scenario with six static landmarks with uncertain positions and a robot equipped with a range-only sensor aiming to minimize uncertainty in the landmark position distribution (see Fig. 2). The robot follows differential-drive dynamics discretized with sampling period τ:

$$\begin{bmatrix} x^1_{t+1} \\ x^2_{t+1} \\ \theta_{t+1} \end{bmatrix} = \begin{bmatrix} x^1_t \\ x^2_t \\ \theta_t \end{bmatrix} + \begin{bmatrix} \nu\tau \operatorname{sinc}(\frac{\omega\tau}{2}) \cos(\theta_t + \frac{\omega\tau}{2}) \\ \nu\tau \operatorname{sinc}(\frac{\omega\tau}{2}) \sin(\theta_t + \frac{\omega\tau}{2}) \\ \tau\omega \end{bmatrix} \tag{33}$$

The action space is composed of discretized control inputs: $\{(\nu, \omega) | \ \nu \in \{1, 3\}$ m/s, $\omega \in \{0, \pm 1, \pm 3\}$ rad/s$\}$.

The target state has dimension $n = 12$ as it is composed of the 2D landmark locations. The targets are assumed static:

$$y_{t+1} = y_t \tag{34}$$

Any of the previously described sensing models $c(x, y)$, corresponding to equations (15), (17), (23), or (26) could be used. Our example illustrates the behavior for a range sensor (17) with measurement noise standard deviation of $\sigma_r = 0.15$ m.

The joint measurement space consists of possible measurements for each landmark $m \in \{0, ...M-1\}$, where M is the total number of landmarks being mapped. The linearized observation model for the joint target state can then be expressed as a block diagonal matrix,

$$C(x, y) = \begin{bmatrix} \nabla_y c(x, y_0) & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \nabla_y c(x, y_{M-1}) \end{bmatrix} \tag{35}$$

Similarly, the bounding sensor information matrix $\bar{M}^t(\cdot)$ is the block diagonal matrix consisting of the blocks of each individual sensor information matrix for each target.

In the simulations, we let the sampling period be $\tau = 0.5$, which means the maximum displacement from any action $r_{\mathcal{U}}$ = 3 m/s × 0.5 (s) = 1.5 meters. Then the reachable set approximations are constructed from Equation 12. The maximum sensing range for the range sensor is 1 meter, with an omnidirectional field of view, which makes the reachable-observable set simple to evaluate given a target distribution. In cases with a limited field of view, the observable set can be over-approximated by an omnidirectional field of view for faster heuristic computation while still ensuring an upper bound on the sensor information matrix.

In Figure 3 we visualize the range sensing heuristic generated with a cost and heuristic based on the trace function: $J_T = \sum_{t=1}^T tr(\Sigma_t)$, for $T = 12$ timestep trajectories. The trace cost is used in the active mapping problem because it is less prone to leaving a target unobserved. Under determinant cost objectives, it is possible to find a path which localizes one target very well drastically reducing its minimum eigenvalue and thus the volume of the confidence ellipsoid, while other targets go unobserved.

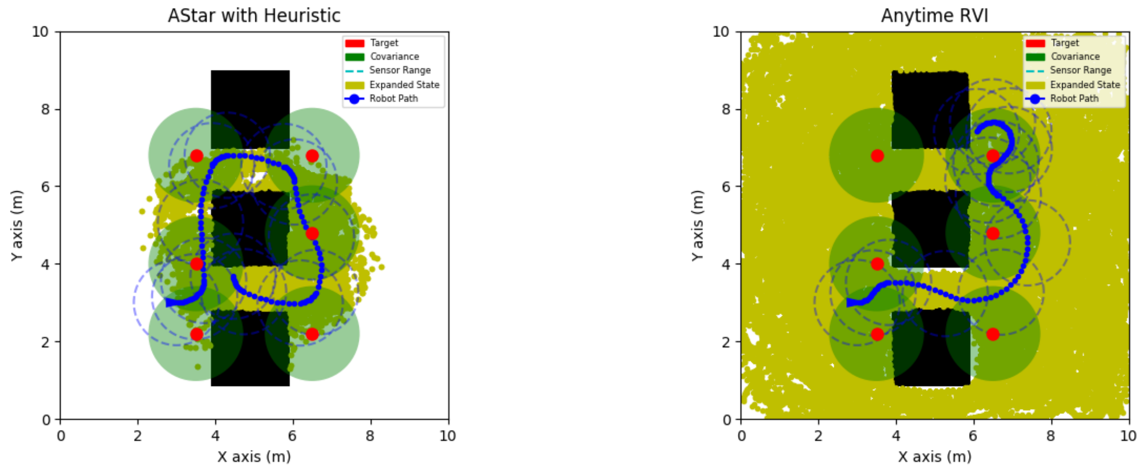
We simulate the six target active mapping task, while varying the initial location of the sensing robot 100 times across the obstacle free space. To compare performance of the two algorithms, we consider the average cost of a node in the search tree generated by each algorithm. In Figure 4 we plot this distribution for the cost function $J = \sum_{t=1}^T trace(\Sigma_t)$. A lower score on this metric indicates that an average node taken from a given search tree has a more optimal score, and thus the search tree has been constructed more effectively.

The heuristic cost map clearly demonstrates the idea that states whose t-step reachable sets may observe the target given the reachable set and observable set approximations, have a much lower heuristic value. The more targets that are observable from a given state, the smaller the remaining cost-to-go in a given state.

To visualize the resulting paths output from the A* algorithm with our proposed heuristic, we plot the planned paths for a single sensor tasked with localizing six landmarks. The prior uncertainty for each target is given as $\Sigma_0 = .25 I_2$. The environment is a 10 meter by 10 meter region, containing three rectangular obstacles in the center. We visualize the sensing robot's planned trajectory and chosen observation points, along with the set of states which have been expanded during the construction of the search tree in Figure 2 (a).

B. Analysis

In Figure 2 (b) we compare our method with the Anytime Reduced Value Iteration algorithm, which is another algorithm for solving information acquisition problems of the form we proposed in [25]. That algorithm does not use a heuristic when constructing the search tree, but instead attempts to iteratively construct the tree in a breadth first manner, pruning any nodes which satisfy a domination



(a) Expanded search tree with the proposed A* heuristic method.

(b) Expanded search tree for Anytime Reduced Value Iteration.

Fig. 2: State expansions resulting from an A* algorithm using the proposed information acquisition heuristic (a) and Anytime Reduced Value Iteration algorithm [25] (b). Note that in (a), the state expansions are focused only in the areas where there are possible observations to be obtained about a landmark. In contrast, figure (b) shows state expansions covering the whole state space. Despite expanding more densely, the algorithm gets stuck in local minima observing only three of the six possible landmarks.

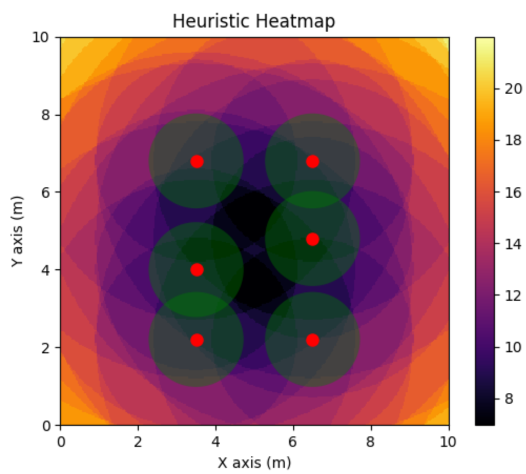


Fig. 3: Heatmap of the heuristic function for the range-sensor, showing the estimated cost-to-go over the state space.

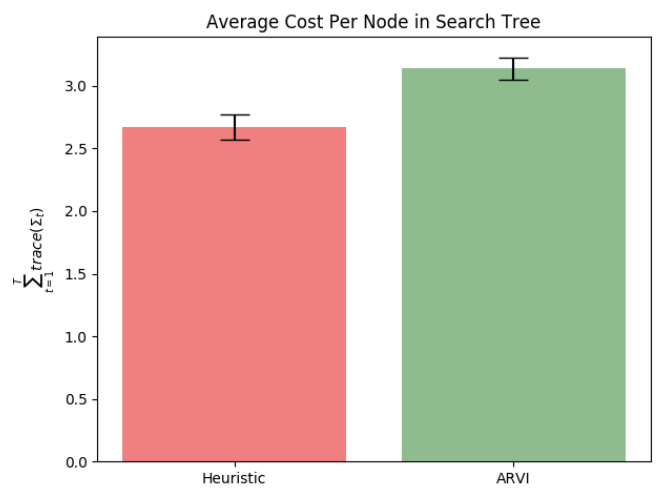


Fig. 4: Graph showing the distribution of the average cost per path in the search tree for both A* (Mean=2.67) and ARVI (Mean=3.14).

criteria, which removes nodes which early in the search are deemed unlikely to lead to an optimal path. The tree is built by expanding non-dominated node, in contrast to the A* algorithm which expands only the node with the most promise.

The results in Figure 2 depict a scenario where a sensing robot with the dynamics and sensing models described in the previous section needs to navigate around the obstacles to localize six landmarks. The A* method based on our proposed heuristic is able to find an optimal path which observes all six landmarks in less than 10 seconds, while the Anytime RVI algorithm fails to find a path that observes all the targets quickly enough, despite being given 60 seconds of execution time on a 2.4 GHz quad core CPU. This improved success in the active mapping task is due to the heuristic which intelligently constructs the search tree in areas where there are possible observations, while the ARVI algorithm spends too much search time in uninformative areas. The algorithms and heuristics are implemented with open source C++ and Python bindings at `https://bitbucket.org/brentsc/infoplanner/src/iros_2019/`.

The graph in Figure 4 shows that the average cost per node in the A* generated search tree is larger than the average cost of nodes in the ARVI generated search tree. This demonstrates quantitatively that the search tree is constructed more effectively, since on average it contains a set of nodes that lead to better trajectories for the sensing robot. We note here however, that due to the continuous motion model, the actual number of nodes can grow quite large in a given horizon. With an action space of size $\|\mathcal{U}\| = 10$, the search tree can grow in $T = 12$ steps to 10^{12}, or one trillion possible states. The heuristic method drastically reduces the search space to make this tractable, but future work may include methods to join states that cross within a given radius δ via the ideas described in [12], [13], which results in smaller overall tree size.

VII. Conclusion

In this work, we formulate the information acquisition problem as a shortest path planning problem, and solved it with A*. To do this efficiently, we proposed the first consistent and admissible heuristic, which guarantees an optimal A* algorithm. Computation of the heuristic requires bounds on the sensor information matrix for the given observation model, which we derive for position, range, bearing, and camera sensing models. The simulation results demonstrate the search space is explored more efficiently than existing methods. Despite the benefits of the heuristic planning method demonstrated here, the heuristic offers little advantage when information is very dense in space, or the heuristic is otherwise unable to guide the search in very ambiguous cases. Furthermore, with continuous motion models for the sensors, the size of the search tree can grow very large since previous states will not be visited again exactly. To alleviate these issues, future work can combine domination criteria with a heuristic, which may lead to an algorithm suitable for a more diverse set of environments. Additional work may include deriving heuristics for more types of sensors, and extending the heuristic method to multi-robot planning.

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