Restoration of Multi-channel Spectral Estimation Affected by Sampling Jitters

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Abstract– **Sampling jitters can cause phase shifts in the Fourier coefficients of an underlying time series and lead to, among other distortions, attenuation in spectral estimation. If the probability density function of sampling jitters is known, then the attenuation can be modeled as an amplitude modulation in the frequency domain. This paper examines the possible effects of random sampling jitters on multi-channel power spectral estimation and discusses a restoration procedure to remove these effects. Simulation tests were conducted to illustrate the effects due to sampling jitters and to demonstrate the effectiveness of the restoration procedure.**

I. INTRODUCTION

Unlike deterministic and periodic sampling, stochastic sampling has the sampling instants subject to random uncertainties (known as jitters). Such situations are not uncommon in applications, e.g., biomedical sensors subject to body motion and sonar sensor arrays to water movement. Most existing research activities related to this topic are focused on reconstruction of data samples at regular intervals, [2]-[4].

It can be shown through Fourier analysis that sampling jitters in sampling locations lead to phase shifts in Fourier coefficients [1],[5]. Furthermore, it can be demonstrated too that the first order moment of Fourier coefficients affected by sampling jitters is modulated by a monotonically decreasing curve in frequency.

This paper examines the effects of sampling jitters on multi-channel spectral estimation. It is shown that if random jitters of all channels are mutually independent then the effects can be treated channel by channel separately, and the magnitude squared coherence (MSC) function will not be affected. On the other hand, if the sampling jitters are correlated between channels, then MSC will be affected as well. If the statistics of sampling jitters is known, a demodulation procedure can be performed to restore multi-channel spectral estimation.

II. METHODS

A stationary time series is usually sampled with a fixed interval T. With the presence of sampling jitter, data samples are noted as $x(t_n)=x(nT+\delta_n)$ where δ_n is the sampling jitter, i.e., samples are displaced from uniformly spaced sampling intervals. Without loss of generality, we'll consider only the dual-channel case. Through Fourier analysis, data samples subject to random jitters can be defined by

$$\begin{bmatrix} x_1(nT+\delta_{1n}) \\ x_2(nT+\delta_{2n}) \end{bmatrix} = \begin{bmatrix} \int X_1(\omega) e^{j\omega(nT+\delta_{1n})} d\omega \\ \int X_2(\omega) e^{j\omega(nT+\delta_{2n})} d\omega \end{bmatrix}, \qquad (1)$$

where $X_i(\omega)$ $(i=1,2)$ is the Fourier transform of $x_i(t)$.

If the statistics of random jitters $\{\delta_i\}$ is known, then the statistics of (1) can be estimated. For example, the first order moment of (1) is given below,

$$\begin{bmatrix} \int X_1(\omega) e^{j\omega nT} E_{/\delta_{1n}}[e^{j\omega\delta_{1n}}] d\omega \\ \int X_2(\omega) e^{j\omega nT} E_{/\delta_{2n}}[e^{j\omega\delta_{2n}}] d\omega \end{bmatrix} = \begin{bmatrix} \int \mu X_1(\omega) e^{j\omega nT} d\omega \\ \int \mu X_2(\omega) e^{j\omega nT} d\omega \end{bmatrix} \qquad (2)$$

In a case of Gaussian sampling jitters, $\delta_n \sim N(0,\sigma)$, it can be shown that

$$E_{/\delta_n}[e^{j\omega\delta_n}] = C(\omega,\sigma) = e^{-\omega^2\sigma^2/2}$$

Restoration of the power spectrum can be performed by demodulation given below,

$$X(\omega) = \frac{\mu X(\omega)}{C(\omega,\sigma)}. \qquad (3)$$

$\mu X(\omega)$ is the mean value of Fourier transform of $\{x(t_n)\}$. The effects on the cross-spectrum can be observed by examining the following.

$$
\begin{aligned}
&E_{/\delta_{1n},\delta_{2n}}[x_1(t_n)x_2^{*}(t_m)] \\
&= E_{/\delta}[\int X_1(\omega_1)e^{j\omega_1 t_n}d\omega_1 \int X^{*}{}_2(\omega_2)e^{-j\omega_2 t_m}d\omega_2] \qquad (4) \\
&= \int S_{X_1X_2}(\omega)e^{j\omega(n-m)T}E_{/\delta}[e^{j\omega(\delta_{1n}-\delta_{2m})}]d\omega
\end{aligned}
$$

$S_{X1X2}(\omega)$ is the cross-spectrum. If $\{\delta_{1n},\delta_{2n}\}$ are zero mean and jointly Gaussian with variance (σ_1^2,σ_2^2) and a correlation coefficient *r*, then

$$E_{/\delta}[e^{j\omega(\delta_{1n}-\delta_{2m})}] = e^{-(\sigma_1^2+2r\sigma_1\sigma_2+\sigma_2^2)\omega^2/2}. \qquad (5)$$

On the other hand, the magnitude coherence (MSC) function will not be affected if *r=0*.

$$MSC(\omega) = \frac{|S_{X_1X_2}(\omega)|}{\sqrt{S_{X_1}(\omega)S_{X_2}(\omega)}} \qquad (6)$$

III. RESULTS AND DISCUSSION

To illustrate the said effects caused by sampling jitters, a dual-channel simulation example was used. Both channels include two unit-amplitude sinusoids (7 and 11 Hz). Fifty data segments of 128 samples were generated using a sampling frequency of 64 Hz. The sampling jitters are i.i.d. Gaussian random variables with zero mean and standard deviations 0.01 and 0.015, respectively.

Power spectra of the dual-channel time series are estimated by averaging fifty data segments. The expected results in (2) are shown in Fig.1. It's clear that high frequency components suffer more attenuation. Fig.2 shows the restored power spectrum estimates after demodulation in (3).

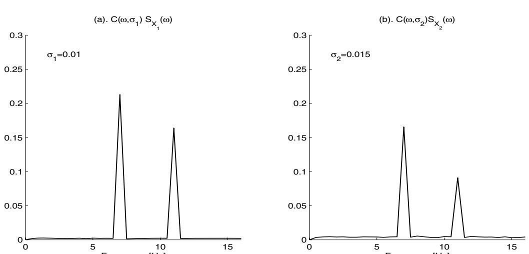

Figure 1: Power spectrum estimation of data subject to sampling jitters. (a) $C(\omega,\sigma)S_{X1}(\omega)$ and δ_{1n} ~N(0,σ=0.01) and (b) $C(\omega,\sigma)S_{X2}(\omega)$ with δ_{2n}~N(0,σ=0.01).

Fig.3 illustrates the effects on cross-spectrum estimation and MSC. The cross-spectrum is a result of modulation delineated in (5). Demodulation can be performed when the statistics of sampling jitters is known.

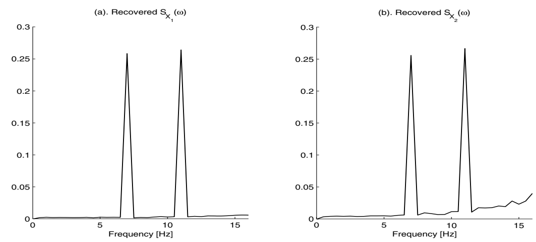

Figure 2: Restored power spectrum estimation of data subject to sampling jitters, (a) $S_{X1}(\omega)$ and (b) $S_{X2}(\omega)$.

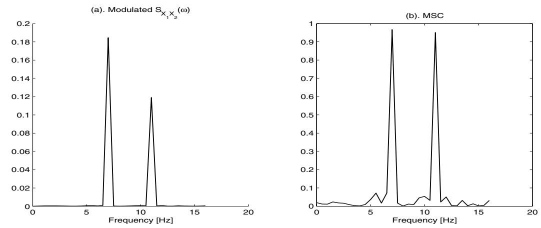

Figure 3: Estimation of dual-channel (a) Modulated Cross-Spectrum $S_{X1X2}(\omega)$, and (b) MSC(ω).

IV. ONCLUSIONS

Sampling jitters are sometimes unavoidable in certain applications. Theoretical analysis on this topic is scarce in literature. This paper uses Fourier analysis and probability theory to illustrate multi-channel sampling jitters and their effects on spectral estimation. It is shown that the power spectrum is unevenly modulated in the frequency domain due to sampling jitters. Demodulation is possible when the probability density function of sampling jitters is known and many data segments are available.

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